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Remarks on Type I Baer and Baer *-Rings

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This paper is concerned with Kaplansky's theory of Baer and Baer *-rings, specifically with the problem of classifying type I factors. We use the terminology in Kaplansky's book [3].

Ornstein [9] calls the pair of dual vector spaces V, W *splittable* if in either V or W every closed subspace admits a closed complement such that the annihilators span the other space. An equivalent formulation is this: for every closed subspace M of V there exists an idempotent-with-adjoint E such that $(V)E = M$. Kaplansky remarks that, given a dual pair V, W, the ring of all linear operators on V that have adjoints on W is a Baer ring if and only if the pair V, W is splittable [3, p. 5]. Kaplansky further states that a ring which is Baer, Zorn, and has no nilpotent ideals $\neq 0$ is a factor of type I if and only if it is primitive with a minimal one-sided ideal [3, p. 19].

Kaplansky's remarks may be combined to obtain a representation theorem for a class of type I Baer factors. For convenience we combine "Zorn and no nilpotent ideals $\neq 0$" into a single axiom:

(EI) For each nonzero X there exists a Y such that XY is a nonzero idempotent.

The Zorn axiom [3, p. 19] is the same as (EI) except that "nonzero X" is replaced by "nonnilpotent X." So (EI) $\Rightarrow$ Zorn, clearly. Also (EI) $\Rightarrow$ no nilpotent ideals $\neq 0$, because if the right ideal $\mathscr{J}$ satisfied $\mathscr{J}^n = 0$, and, if we had a non-zero X in $\mathscr{J}$, then, by (EI), we could construct a nonzero idempotent $E = XY$, which would then belong to right ideal $\mathscr{J}$, a contradiction since $E^n = E \neq 0$.

Conversely, in a type I Baer factor, Zorn and no nilpotent ideals $\neq 0 \Rightarrow$ (EI).

Let $\mathscr{B}$ be the Baer factor in question. Since $\mathscr{B}$ has no nonzero nilpotent ideals, we may conclude:

(1) $R\mathscr{B}S = 0 \Rightarrow R = 0$ or $S = 0$ [3, p. 15].

(2) If E is a nonzero idempotent, $E\mathscr{B}E$ is a factor [3, p. 18].

(3) If $\mathscr{B}$ is type I, then it has a minimal idempotent [3, p. 18].

Let $X \in \mathscr{B}$ be given $X \neq 0$. Using (3) choose a minimal idempotent F. By (1), $F\mathscr{B}X \neq 0$, so there is a $Z \in \mathscr{B}$ such that $FZX \neq 0$. By (1) again, $(FZX)\mathscr{B}F \neq 0$, so there is an M such that $(FZX)MF \neq 0$. Now the ring $F\mathscr{B}F$ is Baer [3, p. 6], Zorn [3, p. 19] and has only the idempotents $0, F$ (since F is minimal). Therefore $F\mathscr{B}F$ is a division ring [3, p. 19]. So, since $(FZX)MF \neq 0$, we can find N such that $(FZX)\,MF(FNF) = F$. Setting $R = MFN$, we have $FZXRF = F$. Now let $Y = RFZ$, and set $E = XY$. We have $FZE = FZ$ so $E \neq 0$ for otherwise we would have $FZ = 0$, contradicting $FZX \neq 0$. Finally we check directly that $E^2 = E$, so $E = XY$ is the desired nonzero idempotent.

We may then make the following assertion.

For each type I Baer factor $\mathscr{B}$ that satisfies (EI), *there is a division ring k, unique up to isomorphism, and a splittable pair V, W of dual vector spaces over k such that $\mathscr{B}$ is isomorphic to a subring $\mathscr{C}$ of the ring $\mathfrak{A}$ of all k-linear operators on V having adjoints on W, the subring $\mathscr{C}$ containing all the finite-rank operators in $\mathfrak{A}$, and containing, for each closed subspace M of V, an idempotent P with $(V)P = M$.*

Conversely, every such subring $\mathscr{C}$ is a type I Baer factor that satisfies (EI).

We would have a more satisfactory representation were we able to prove $\mathscr{C} = \mathfrak{A}$ under the given hypotheses.

Ornstein constructs a class of examples of splittable dual pairs which seems to include all known ones [9, Section 4]. Insofar as this is the case, the above stated result furnishes all known examples of type I Baer factors that satisfy (EI).

The proof of the underlined assertion is a routine extension of the methods used by Kaplansky [10, Part II, Section 6] and Jacobson [2, Chap. IV, Section 9] to represent primitive rings. We outline a proof that makes systematic use of the (EI) axiom (but alert the reader that some of the individual steps such as the nondegeneracy of the form and the faithfulness of the representation can be proved with less than (EI)).

As in (3) above, we select a minimal idempotent E in $\mathscr{B}$, and, using (EI), verify directly that $k = E\mathscr{B}E$ is a division ring. The sets $V = E\mathscr{B}$, $W = \mathscr{B}E$ are, in a natural way, left and right vector spaces, respectively, over k. If $x = EX \in V$ and $y = YE \in W$, then the expression $(x, y) = EXYE$ defines a bilinear k-valued nondegenerate form on $V \times W$, the nondegeneracy being verified with the help of the following result: In a Baer ring $\mathscr{B}$ satisfying (EI), the left annihilator of a left ideal has form $\mathscr{B}G$, G a unique central idempotent. Letting each $T \in \mathscr{B}$ act on V via right multiplication, we obtain an isomorphism of $\mathscr{B}$ into $\mathfrak{A}$, the one-to-one character of this representation being established with the help of the result mentioned directly above. Finally we show that for each closed subspace M of V, there is an idempotent P in $\mathscr{B}$

such that $(V)P = M$. From this fact it follows that V, W is a splittable pair. Those are the major items of the proof.

According to a theorem of Mackey, the dual pair V, W is splittable whenever $\dim(V) = \dim(W) = \aleph_0$ [6], so the corresponding ring $\mathcal{Cl}$ is always a type I Baer factor satisfying (EI), whatever the underlying division ring k. It is therefore rather easy to be a Baer ring. In contrast, it is apparently rather difficult to be a Baer $*$-ring, as our next two results show.

Let k represent a division ring with involution, let V symbolize a left vector space over k on which is defined a conjugate-bilinear, Hermitian, non-isotropic, k-valued form $(\cdot, \cdot)$, and let $\mathcal{Cl}(V, k)$ stand for the $*$-ring of all k-linear operators-with-adjoints on V. If $\dim(V) = \aleph_0$, *then $\mathcal{Cl}(V, k)$ is never a Baer $*$-ring, whatever the division ring k.*

This result is due essentially to Kaplansky [4], although he did not state it in terms of Baer $*$-rings. Kaplansky's ideas were also used by Morash [7] to get a closely related kind of result bearing on the nonorthomodularity of the lattice of closed subspaces of V. We can in fact relate the above assertion directly to Morash's theorem by the following useful lemma, which is valid for a V of any dimension. For $N \subseteq V$ we write $N^{\perp}$ for the set of $x \in V$ satisfying $(x, N) = 0$. The subspace N is *closed* when $N = N^{\perp\perp}$.

LEMMA. *Let k, V, $(\cdot, \cdot)$ and $\mathcal{Cl}(V, k)$ be as above (no restriction on* $\dim(V)$*), and let $\mathscr{C}$ be any $*$-subring of $\mathcal{Cl}(V, k)$ that contains all the rank-one operators. (In particular we may take $\mathscr{C} = \mathcal{Cl}(V, k)$.) If $\mathscr{C}$ is a Baer $*$-ring, then a subspace N of V has the form $N = (V)F$ for some projection $F \in \mathscr{C}$ if and only if $N = N^{\perp\perp}$.*

Proof. Given a subspace N of V satisfying $N = N^{\perp\perp}$, let $\mathscr{S}$ stand for the set of S in $\mathscr{C}$ such that $(N)S = 0$. Since $\mathscr{C}$ is a Baer $*$-ring, the left annihilator of $\mathscr{S}$ has the form $\mathscr{C}F$, F a projection in $\mathscr{C}$. We shall show $(V)F = N$.

We have $V(F) \subseteq N$, for otherwise there would exist an $x_0 \in V$ such that $y_0 = x_0F \notin N = (N^{\perp})^{\perp}$, which would in turn imply the existence of a $z_0 \in N^{\perp}$ with $(y_0, z_0) \neq 0$. But then the operator S defined by $xS = (x, z_0)\, y_0$, being a rank-one operator-with-adjoint would belong to $\mathscr{C}$ and also to $\mathscr{S}$ because $N(S) = 0$, but yet would not be left-annihilated by F, since $y_0FS = y_0S = (y_0, z_0)\, y_0 \neq 0$, a contradiction.

On the other hand, if $x_0 \in N$, $x_0 \neq 0$, then the projection P defined by $zP = (z, x_0)(x_0, x_0)^{-1}x_0$ clearly lies in the left annihilator of $\mathscr{S}$, so $PF = P$. This means $x_0 = x_0P = x_0PF = x_0F$, so $x_0 \in (V)F$. Hence $N \subseteq (V)F$, so $N = (V)F$ as claimed.

The converse assertion that every N of the form $N = (V)F$ has $N = N^{\perp\perp}$ follows directly from the fact that the kernel of any operator-with-adjoint is a closed subspace ($N = \ker(I - F)$).

The lemma has the following important consequence: the closed subspaces of V comprise an orthomodular lattice when $\mathscr{C}$ is a Baer *-ring [5, p. 50]. The following corollary gives another way of stating this fact (we maintain the notation of the lemma).

COROLLARY. *If $\mathscr{C}$ is a Baer *-ring, then $N + N^{\perp} = V$ for any closed subspace N of V. Thus $B^{\perp} + B^{\perp\perp} = V$ for any subset B of V.*

Proof. Since N is closed, there is a projection $E \in \mathscr{C}$ with $(V)E = N$. Given $x \in V$ we write $x = xE + x(I - E)$. Clearly $xE \in N$. Also $x(I - E) \in N^{\perp}$ as a routine computation shows. That proves $N + N^{\perp} = V$, and the last sentence follows from the observation that $B^{\perp}$ is always a closed subspace of V, whatever the subset B.

Returning now to the proof of the second underlined statement, Kaplansky has shown that it is impossible to have $N + N^{\perp} = V$ for all closed subspaces of V when $\dim(V) = \aleph_0$ [4]; we refer the reader to Morash's paper [7] for the details. Thus, when $\dim(V) = \aleph_0$, all such *-subrings $\mathscr{C}$, including $\mathscr{O}(V, k)$ itself, fail to be Baer *.

Our next result, which is closely related to another theorem of Morash [8], illustrates that the division ring k itself, in addition to the dimension of V over k, may be dramatically restricted by the assumption that $\mathscr{O}(V, k)$ is a Baer *-ring. But to secure this next result we need to invoke Kaplansky's EP axiom [3, Chap. 13] which is analogous to the EI axiom introduced above:

> (EP) For every nonzero X there is a self-adjoint Y such that $Y\,CC\,X^*X$ and X^*XY^2 is a nonzero projection.

THEOREM. *Let k stand for a division subring of the real quaternions that is closed under quaternionic conjugation, and let V represent an infinite-dimensional left vector space over k on which is defined a conjugate-bilinear, Hermitian, positive definite, k-valued form $(\cdot, \cdot)$. Let $\mathscr{O}(V, k)$ symbolize the *-ring of all k-linear operators-with-adjoints on V.*

*If $\mathscr{O}(V, k)$ is a Baer *-ring satisfying EP, then:*

(1) *k is the real numbers, the complex numbers, or the quaternions,*

(2) *V is Hilbert space, and*

(3) *$\mathscr{O}(V, k)$ is the ring of all bounded linear operators on V.*

Proof. By scaling the form, we may assume without loss of generality that there is vector $e \in V$ with $(e, e) = 1$. Then, applying the EP axiom directly to the rank-one operators, we deduce that, given any $f \in V$, $f \neq 0$, there is a real λ such that $(\lambda f, \lambda f) = 1$. That is to say, in any direction there is a vector of length 1. This is the only use we make of the EP axiom. Owing to the infinite dimensionality of V, we can find an infinite orthonormal sequence in V.

The real-valued function $\rho(x, y) = \| x - y \|$ is a metric on the space V; let H represent the completion of V with respect to this metric. H is a k-vector space under the natural extension-by-continuity of V's operations.

The completion $\bar{k}$ of k is either the real numbers, the complex numbers or the quaternions. Given $\alpha \in \bar{k}$ we select $\alpha_n \in k$, $\alpha_n \to \alpha$, and, for any $x \in H$, define

$$\alpha x = \lim \alpha_n x,$$

the sequence on the right being Cauchy in H and therefore necessarily convergent. This definition is effective and, extending the inner product by continuity, we make H a Hilbert space over $\bar{k}$. We regard k as a subset of $\bar{k}$, V as a subset of H, and our objective is to prove $k = \bar{k}$, $V = H$. Professors E. A. Connors and R. P. Morash provided me with crucial help with this proof, and I am very grateful for their assistance. The key idea comes from a paper of Amemiya and Araki [1].

As we have already pointed out, we can prove, using the EP axiom, the existence of an infinite orthonormal sequence in V, say $(e_i;\ i = 1, 2,...)$. If $(\rho_i;\ i = 1, 2,...)$ is a sequence of elements from k such that $\sum_{i=1}^{\infty} | \rho_i |^2 < \infty$, then the vector $a = \sum_{i=1}^{\infty} \rho_i e_i$ belongs to H. I shall show first that, if $(a, a) = \sum_{i=1}^{\infty} | \rho_i |^2 \in k$, if $\rho_i \neq 0$, $i = 1, 2,...$, and if $| \rho_n |^{-1} \sum_{i=n+1}^{\infty} | \rho_i |^2 \to 0$ as $n \to \infty$, then $a \in V$.

Fix such an $a = \sum_{i=1}^{\infty} \rho_i e_i$, taking, without loss of generality, $\rho_1 = 1$. Set $b = \beta e_1 - \sum_{i=2}^{\infty} \rho_i e_i$ where $\beta = \sum_{i=2}^{\infty} | \rho_i |^2$. We have $\beta = (a, a) - 1 \in k$. One checks easily that $(a, b) = 0$ and that $a + b = (\beta + 1)\, e_1 = (a, a) e_1 \in V$.

A straightforward calculation reveals that the sequences

$$a_n = \sum_{i=1}^{2n-1} \rho_i e_i + \left(\beta - \sum_{i=2}^{2n-1} | \rho_i |^2\right) \bar{\rho}_{2n}^{-1} e_{2n},$$

and

$$b_m = \beta e_1 - \sum_{i=2}^{2m} \rho_i e_i - \left(\beta - \sum_{i=2}^{2m} | \rho_i |^2\right) \bar{\rho}_{2m+1}^{-1} e_{2m+1},$$

$n, m = 1, 2,...$, have the following properties:

$$a_n, b_m \in V, \qquad n, m = 1, 2,...,$$

$$(a_n, b) = (a, b_m) = (a_n, b_m) = 0, \qquad n, m = 1, 2,...,$$

$$\| a_n - a \|^2 = \sum_{i=2n+1}^{\infty} | \rho_i |^2 + \left(| \rho_{2n} |^{-1} \sum_{i=2n+1}^{\infty} | \rho_i |^2\right)^2 \to 0,$$

and

$$\| b_m - b \|^2 = \sum_{i=2m+2}^{\infty} | \rho_i |^2 + \left(| \rho_{2m+1} |^{-1} \sum_{i=2m+2}^{\infty} | \rho_i |^2 \right)^2 \to 0.$$

Let B be the subset $\{b_1, b_2, \ldots\}$ of V, and set $N = B^{\perp}$ (orthocomplement taken in V). Then N is a closed subspace of V, $a_i \in N$, $i = 1, 2, \ldots$, $B \subseteq N^{\perp}$, and, using the corollary to the lemma, $N + N^{\perp} = V$. Accordingly

$$(a, a)\, e_1 = p + q, \qquad p \in N \subseteq V, \qquad q \in N^{\perp} \subseteq V.$$

We construct a metrically closed subspace C of the Hilbert space H by constructing first the subspace C_1 of H generated algebraically by the subset N, then take C as the metric closure of C_1. We construct another closed subspace D from $N^{\perp}$ in the same way. Since orthogonality is preserved at both stages of the construction, we have $C \perp D$. Also $a \in C$, $b \in D$ so $(a, a)\, e_1 = a + b$, $a \in C$, $b \in D$. But we have already shown $(a, a)\, e_1 = p + q$, $p \in N \subseteq C$, $q \in N^{\perp} \subseteq D$, so, by uniqueness, $a = p \in V$, which was to be proved.

Hence V contains every $a = \sum_{i=1}^{\infty} \rho_i e_i$ having the three properties listed, and, inasmuch as our bilinear form is k-valued, k must contain every number (a, b), where a, b are vectors in V of the type described. Since k contains all rationals, the vectors $a = \sum_{i=1}^{\infty} 2^{-i} e_i$, $b = \sum_{i=1}^{\infty} \epsilon_i 2^{-i} e_i$, where $\epsilon_i = \pm 1$, belong to V, because $(a, a) = (b, b) = 1/3 \in k$, and

$$2^n \sum_{i=n+1}^{\infty} | 2^{-i} |^2 = \tfrac{1}{3}(\tfrac{1}{2})^n \to 0.$$

Hence k contains all

$$(a, b) = \sum_{i=1}^{\infty} \epsilon_i 4^{-i}$$

for any choice $\epsilon_1, \epsilon_2, \ldots$, of 1's and -1's. From such numbers, using rational linear combinations, we can obtain any base 4 representation and therefore any real number. Hence k contains the real numbers and is therefore the reals, the complex numbers, or the quaternions.

Therefore our original space V was a classical real, complex, or quaternionic inner product space to begin with. Since its lattice of $\perp$-closed subspaces is orthomodular, it must be complete—this is the original result of Amemiya and Araki [1]. Thus V is a classical Hilbert space. This proves items (1) and (2) of the theorem, and (3) follows from the closed graph theorem.

These results indicate that the Baer * property and infinite dimensionality, taken together, comprise an extremely strong pair of conditions on a *-ring, and indicate further that, with the help of these conditions, it may be possible to prove some rather precise classification theorems for *-rings. Specifically, one can set the following problem (using Kaplansky's terminology):

Find all infinite type I *Baer *-factors that satisfy the* EP *axiom.*

The only such *-rings I know are the three classical rings described in the theorem.

A similar lattice theory classification problem has been outlined by Morash [8]. The lattice problem and the ring problem are closely related.

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